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EIGENVALUES AND ENERGY OF ARITHMETIC FUNCTION GRAPH OF A FINITE GROUP

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Abstract. Given an arithmetical function h, the arithmetic function graph $G_h(\mathfrak{G})$ of a finite group $\mathfrak{G}$ with respect to h is defined as a graph with vertex set $V(G_h(\mathfrak{G})) = \mathfrak{G}$ and any two distinct vertices a and b are adjacent in $G_h(\mathfrak{G})$ if and only if $h(|a||b|) = h(|a|)h(|b|)$. The energy of a graph is the sum of the absolute values of the eigenvalues of the adjacency matrix of the graph. In this paper, we discuss some results on eigenvalues and energy of arithmetic function graphs of finite groups.

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1. Introduction

For standard terminology and notion in group theory and graph theory, we refer the reader to the text-books of Harary [9] and Herstein [10] respectively. The non-standard will be given in this paper as and when required.

Throughout this paper, $\mathfrak{G}$ denotes a finite group. The order of an element in a group $\mathfrak{G}$ is denoted by $|a|$ and order of $\mathfrak{G}$ is denoted by $o(\mathfrak{G})$. The greatest common divisor (gcd) of two numbers x and y is denoted by (x, y) and n denotes a positive integer.

There are many concepts related to chemistry involving groups and graphs. This motivated us to study eigen values and energy of arithmetic function graphs. In this paper, we discuss some results on eigenvalues and energy of Arithmetic function graphs of finite groups.

We recall the following basic definitions and results:
Let G be a graph with n vertices $v_1, v_2, \ldots, v_n$. The adjacency matrix of G, denoted by $A = A(G)$, is a square matrix of order n whose (i, j)-entry is defined as:

$$A_{ij} = \begin{cases} 1, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of $A(G)$ are said to be the eigenvalues of the graph G. The spectrum of G is the collection of eigen values of G. We denote largest

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and smallest eigenvalues of a graph G by λ_{max} and λ_{min} respectively. If H is a subgraph of a graph G, we have

$$\lambda_{max}(G) \geq \lambda_{max}(H). \tag{1}$$

For any graph G,

$$\lambda_{max} \leq d_{max}, \tag{2}$$

where d_{max} is the maximum vertex degree of G (See [1]).

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of a graph G. The energy of G (See [1, 6–8]) is defined as:

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|. \tag{3}$$

Energy of the complete graph K_n is

$$\mathcal{E}(K_n) = 2n - 2. \tag{4}$$

Energy of the complete bipartite graph $K_{1,n-1}$ is

$$\mathcal{E}(K_{1,n-1}) = 2\sqrt{n-1}. \tag{5}$$

For more results on eigenvalues and energy associated with graphs, we suggest the reader to refer the papers [11–14, 16, 24, 25].

The concept of order prime graph was introduced by M. Sattanathan and R. Kala [23]. Further, Ma et al. [15] and Dorbidi [5] have studied the order prime graphs (coprime graphs) of finite groups. The order prime graph $OP(\mathfrak{G})$ of a finite group $\mathfrak{G}$ is a graph with the vertex set $V(OP(\mathfrak{G})) = \mathfrak{G}$ and two distinct vertices a and b are adjacent in $OP(\mathfrak{G})$ if and only if $(o(a), o(b)) = 1$.

The concept of general order prime graph was introduced by R. Rajendra and P. S. K. Reddy [17, 18]. The general order prime graph $GOP(\mathfrak{G})$ of a finite group of order n is a graph with the vertex set $V(GOP(\mathfrak{G})) = \mathfrak{G}$ and two vertices a and b are adjacent in $GOP(\mathfrak{G})$ if and only if $(o(a), o(b)) = 1$ or p, where p is a prime and $p < n$, and $a \neq b$. Clearly, $G_\phi(\mathfrak{G})$ is a subgraph of $GOP(\mathfrak{G})$.

We have introduced the concept of arithmetic function graph of a finite group in [22] and investigated some results. The Arithmetic function graph $G_h(\mathfrak{G})$ of $\mathfrak{G}$ with respect to an arithmetical function h is defined as a graph with vertex set $V(G_h(\mathfrak{G})) = \mathfrak{G}$ and two distinct vertices a and b are adjacent in $G_h(\mathfrak{G})$ if and only if $h(|a||b|) = h(|a|)h(|b|)$. We have observed that the order prime graph of a finite group $OP(\mathfrak{G})$ is nothing but the arithmetic function graph $G_\phi(\mathfrak{G})$ with respect to the Euler's ϕ-function. Also, it is proved that $G_\phi(\mathfrak{G})$ is a subgroup of $G_h(\mathfrak{G})$, for any multiplicative function h.

2. Results

Theorem 2.1. *If $\mathfrak{G}$ is a group of order n and h is a multiplicative function, then*

$$max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} \leq \lambda_{max}(G_\phi(\mathfrak{G})) \leq \lambda_{max}(G_h(\mathfrak{G})) \leq n-1. \tag{6}$$

In particular, if $n \geq 3$, $\sqrt{n-1} \leq \lambda_{max}(G_\phi(\mathfrak{G})) \leq \lambda_{max}(G_h(\mathfrak{G})) \leq n-1$.

Proof. Suppose that $\mathfrak{G}$ is a group of order n and h is a multiplicative function. Since h is multiplicative, $G_\phi(\mathfrak{G})$ is a subgraph of $G_h(\mathfrak{G})$ and hence from (1) it follows that

$$\lambda_{max}(G_h(\mathfrak{G})) \geq \lambda_{max}(G_\phi(\mathfrak{G})). \tag{7}$$

From (2), we have

$$\lambda_{max}(G_h(\mathfrak{G})) \leq n-1. \tag{8}$$

Also, by [19, Theorem 1], we have

$$max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} \leq \lambda_{max}(G_\phi(\mathfrak{G})) \leq n-1. \tag{9}$$

Then (6) follows from (7), (8) and (9).
Clearly, for $n \geq 3$,

$$max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} = \sqrt{n-1}$$

and from (6), it follows that,

$$\sqrt{n-1} \leq \lambda_{max}(G_\phi(\mathfrak{G})) \leq \lambda_{max}(G_h(\mathfrak{G})) \leq n-1.$$ □

Corollary 2.2. *Let $\mathfrak{G}$ be a finite group of order $n \leq 2$ and h be a multiplicative function. Then*

(10)

$$\lambda_{max}(G_\phi(\mathfrak{G})) = \lambda_{max}(G_h(\mathfrak{G})) = \lambda_{max}(GOP(\mathfrak{G})) = \begin{cases} 0, & \text{for } n=1; \\ 1, & \text{for } n=2. \end{cases}$$

Proof. By [21, Theorem 2.1], we have

$$max\left\{\frac{2n-2}{n},\ \sqrt{n-1}\right\} \leq \lambda_{max}(G_\phi(\mathfrak{G})) \leq \lambda_{max}(GOP(\mathfrak{G})) \leq n-1. \tag{11}$$

From (6) and (11), equation (10) follows. □

Note: If $\mathfrak{G}$ is a finite group of order 2, then for any arithmetical function h with $h(1) = 1$, we have $G_\phi(\mathfrak{G}) = GOP(\mathfrak{G}) = G_h(\mathfrak{G}) \cong K_2$, the complete graph on two vertices and consequently, $\lambda_{min}(G_h(\mathfrak{G})) = -1$ and $\lambda_{max}(G_h(\mathfrak{G})) = 1$.

Theorem 2.3. *Let $\mathfrak{G}$ be a finite group of order $p > 2$, where p is a prime, and h be a multiplicative function. Then*

- *(i) $h(p^2) \neq h(p)^2$ if and only if for each eigenvalue λ of $G_h(\mathfrak{G})$, $-\lambda$ is an eigenvalue with the same multiplicity.*
- *(ii) $h(p^2) \neq h(p)^2$ if and only if*

$$\lambda_{min}(G_h(\mathfrak{G})) = -\lambda_{max}(G_h(\mathfrak{G})).$$

Proof. By [22, Theorem 4.13(i)], we have $G_h(\mathfrak{G})$ is a star if and only if $h(p^2) \neq h(p)^2$. Hence by [2, Proposition 3.4.1, p.38], the proof of (i) and (ii) follows. □

Since $OP(\mathfrak{G}) = G_\phi(\mathfrak{G})$, we have the following result from [19]:

Theorem 2.4. *[19, Theorem 3] Let $\mathfrak{G}$ be a finite group of order $n \geq 3$, then $G_\phi(\mathfrak{G})$ has at least three distinct eignvalues.*

Theorem 2.5. *Let h be a multiplicative function such that $h(r^2) \neq h(r)^2$ for any prime r. If $\mathfrak{G}$ is a non-abelian group of order pq, where p and q are distinct primes with $p < q$, then $G_h(\mathfrak{G})$ has at least three distinct eignvalues.*

Proof. A connected graph with diameter d, has at least $d+1$ distinct eigenvalues [2, Proposition 1.3.3, p.5]. By [22, Theorem 4.19(iii)], $G_h(\mathfrak{G})$ is non-planar of diameter 2. Hence it follows that $G_h(\mathfrak{G})$ has at least three distinct eignvalues. □

Definition 2.1. *Let $\mathfrak{G}$ be a group of finite order and h be an arithmetical function. The h-energy of $\mathfrak{G}$, denoted by $\mathcal{E}_h(\mathfrak{G})$, is defined as the energy of the arithmetic function graph $G_h(\mathfrak{G})$. That is, $\mathcal{E}_h(\mathfrak{G}) = \mathcal{E}(G_h(\mathfrak{G}))$.*

Proposition 2.6. *If $\mathfrak{G}$ is a group of order n and h is a completely multiplicative function, then $\mathcal{E}_h(\mathfrak{G}) = 2n - 2$.*

Proof. For a completely multiplicative function h, $G_h(\mathfrak{G}) \cong K_n$. Therefore from (4), we have $\mathcal{E}_h(\mathfrak{G}) = 2n - 2$. □

Proposition 2.7. *If $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are isomorphic finite groups, then $\mathcal{E}_h(\mathfrak{G}_1) = \mathcal{E}_h(\mathfrak{G}_2)$ for any arithmetical function h.*

Proof. If $\mathfrak{G}_1 \cong \mathfrak{G}_2$, then by [22, Proposition 4.3], for any arithmetical function h we have $G_h(\mathfrak{G}_1) \cong G_h(\mathfrak{G}_2)$ and hence $\mathcal{E}_h(\mathfrak{G}_1) = \mathcal{E}_h(\mathfrak{G}_2)$. □

Remark 2.8. The converse of the Proposition 2.7 is not true. For instance, for a completely multiplicative function h, we have $G_h(V_4) \cong G_h(\mathbb{Z}_4) \cong K_4$ and hence $\mathcal{E}_h(V_4) = \mathcal{E}_h(\mathbb{Z}_4) = \mathcal{E}(K_4) = 2 \cdot 4 - 2 = 6$ (from (4)); but the groups V_4 and $\mathbb{Z}_4$ are not isomorphic.

Since $OP(\mathfrak{G}) = G_\phi(\mathfrak{G})$, we have the following result from [19]:

Theorem 2.9. *[19, Theorem 5] Let $\mathfrak{G}$ be a finite group.*

(i) If $o(\mathfrak{G}) = 2$, then $G_\phi(\mathfrak{G}) \cong K_2$ and $\mathcal{E}_\phi(\mathfrak{G}) = 2$.

(ii) If $o(\mathfrak{G}) = n$, then

$$\mathcal{E}_\phi(\mathfrak{G}) \leq \frac{n(\sqrt{n}+1)}{2}. \tag{12}$$

(iii) If $o(\mathfrak{G}) = n = p^\alpha$ where p is a prime and $\alpha \in \mathbb{Z}^+$, then

$$.\mathcal{E}_\phi(\mathfrak{G}) \leq \frac{n(\sqrt{n}+\sqrt{2})}{\sqrt{8}}. \tag{13}$$

(iv) If $o(\mathfrak{G}) = n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ where p_i's are primes and $n_i \in \mathbb{Z}^+, \forall i$, then

$$\mathcal{E}_\phi(\mathfrak{G}) \leq \sqrt{n\left(n^2 - n + \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} x_i^2\right)}, \tag{14}$$

where x_i is the number of elements in $\mathfrak{G}$ of order p_i^j, $1 \le j \le n_i$, $1 \le i \le k$.

Theorem 2.10. *Let $\mathfrak{G}$ be a finite group of order $p > 2$, where p is a prime, and h be a multiplicative function. Then*

$$\mathcal{E}_h(\mathfrak{G}) = \begin{cases} 2\sqrt{n-1}, & \text{if } h(p^2) \neq h(p)^2; \\ 2(p-1), & \text{if } h(p^2) = h(p)^2. \end{cases}$$

Proof. By [22, Theorem 4.13], we have

(i) $G_h(\mathfrak{G})$ is a star if and only if $h(p^2) \neq h(p)^2$,

(ii) $G_h(\mathfrak{G})$ is a complete graph if and only if $h(p^2) = h(p)^2$.

Then from (4) and (5), the proof follows. □

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References

[1] R. B. Bapat, *Graphs and matrices*, Universitext, Springer, 2010.

[2] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs - Monograph*, Springer, 2011.

[3] D. M. Burton, *Elementary number theory*, 7th ed., McGraw Hill Education (India), 2012.

[4] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, 1979.

[5] H. R. Dorbidi, A note on the coprime graph of a group, *Int. J. Group Theory*, 5(4) (2016), 17–22.

[6] I. Gutman, The energy of a graph, *Ber. Math.-Statist. Sekt. Forschungsz. Graz*, 103 (1978), 1–22.

[7] I. Gutman, The Energy of a Graph: Old and New Results. In: Betten, A., Kohnert, A., Laue, R. and Wassermann, A., Eds., *Algebraic Combinatorics and Applications*, Springer, Berlin, (2001), 196–211.

[8] I. Gutman et al., On the energy of regular graphs, *MATCH Commun. Math, Comput. Chem.*, 57 (2007), 435–442.

[9] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, 1972.

[10] I. N. Herstein, *Topics in Algebra*, Second Ed., John Wiley & Sons, 2003.

[11] V. Lokesha, Y. Shanthakumari and P. Siva Kota Reddy, Skew-Zagreb Energy of Directed Graphs, *Proceedings of the Jangjeon Math. Soc.*, 23(4) (2020), 557-568.

[12] K. N Prakasha, P. Siva Kota Reddy and I. N. Cangul, Partition Laplacian Energy of a Graph, *Advn. Stud. Contemp. Math.*, 27(4) (2017), 477-494.

[13] K. N Prakasha, P. Siva Kota Reddy and I. N. Cangul, Minimum Covering Randic energy of a graph, *Kyungpook Mathematical Journal*, 57(4) (2017), 701-709.

[14] K. N Prakasha, P. Siva Kota Reddy and I. N. Cangul, Sum-Connectivity Energy of Graphs, *Adv. Math. Sci. Appl.*, 28(1) (2019), 85-98.

[15] X. Ma, H. Wei and L. Yang, The coprime graph of a group, *Int. J. Group Theory*, 3(3) (2014), 13–23.

[16] K. V. Madhusudhan, P. Siva Kota Reddy and K. R. Rajanna, Randic type Additive connectivity Energy of a Graph, *Vladikavkaz Mathematical Journal*, 21(2) (2019), 18-26.

[17] R. Rajendra and P. Siva Kota Reddy, On general order prime graph of a finite group, *Proc. Jangjeon Math. Soc.*, 17(4) (2014), 641–644.

[18] R. Rajendra, P. Siva Kota Reddy and V. M. Siddalingaswamy, On general order prime graph of a finite group-II, *Adv. Appl. Discrete Math.*, 17(4) (2016), 437–444.

[19] R. Rajendra, A. C. Chandrashekar and B. M. Chandrashekara, A note on energy of order prime graph of a finite group, *International Journal of Scientific and Engineering Research*, 7(5) (2016), 15–16.

[20] R. Rajendra, P. Siva Kota Reddy and K. V. Madhusudhan, Set-Prime Graph of a Finite Group, *Proc. Jangjeon Math. Soc.*, 22(3) (2019), 455–462.
[21] R. Rajendra and P. S. K. Reddy, Some Results on Order Prime Graphs and General Order Prime graphs, *Proc. Jangjeon Math. Soc.*, 24(2) (2021), 223–230.
[22] R. Rajendra, R. Kemparaju and P. Siva Kota Reddy, Arithmetic Function Graph of a Finite Group, *Palestine Journal of Mathematics*, 11(2) (2022), 488–495.
[23] M. Sattanathan and R. Kala, An introduction to order prime graph, *Int. J. Contemp. Math. Sciences*, 10(4) (2009), 467–474.
[24] P. Siva Kota Reddy, K. N Prakasha and I. N. Cangul, Inverse Sum Indeg Energy of a Graph, *Advn. Stud. Contemp. Math.*, 31(1) (2021), 7-20.
[25] P. Siva Kota Reddy, K. N Prakasha and I. N. Cangul, Randić Type Hadi Energy of a Graph, *The Nepali Mathematical Sciences Report*, 39(2) (2022), 95-105.

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